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Relation Between Classical and Quantum Mechanics†

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Abstract

Expressions for the Lie derivatives of functions of non-commuting variables are derived and used to reformulate classical mechanics. This is possible only if the phase space variables commute, or if they satisfy Heisenberg's commutation relations.

1. *Introduction*

The structural similarity between classical and quantum mechanics provides a useful approach to the study of the mathematical foundations of the two theories. This structural similarity is manifested most conspicuously in the set of dynamical variables: In both theories, the set of dynamical variables is a linear space equipped with two algebraic structures, an associative product, and a Lie product. In classical mechanics, the dynamical variables are represented by real-valued functions of class C^∞ over phase space; the associative product is the ordinary product of such functions, which is also commutative, and the Lie product is the Poisson bracket. In the usual representation of quantum mechanics, the dynamical variables are operators in Hilbert space; the associative product is the composition of such operators, which is not commutative, and the Lie product is essentially the commutator.

The natural starting point for the study of the structural relationship between classical and quantum mechanics is a fundamental uniqueness theorem due to Falk (1951), according to which exactly two associative algebras admit a Lie product satisfying the axioms of the Poisson bracket: The algebra of functions of commuting phase space variables, and the algebra of functions of phase space variables which satisfy Heisenberg's commutation relations.

The problem of introducing a Poisson structure into an associative algebra may be analysed from two points of view. On the one hand, one

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can adopt a purely algebraic attitude, and view the Poisson product as an abstract, axiomatically defined operation. The problem consists in finding all associative algebras which admit such a product, and in exhibiting, in each of these algebras, an algorithm for this product. This problem has been solved elsewhere (Grgin & Petersen, 1972). The algebraic approach also yields a solution to the problem of representing canonical algebras in auxiliary algebras. On the other hand, one can adopt a geometric attitude, and interpret the Poisson product as a Lie derivative. The concept of the Lie derivative is the basic tool for the study of groups of motions which leave invariant some geometric object of interest. In differential geometry one considers, for example, the Killing motions, which preserve the metric tensor, the conformal motions, which preserve the metric tensor density, etc.

In this paper we study the structural relation between classical and quantum mechanics from a point of view suggested by the differential geometric formulation of classical mechanics. For this purpose we first derive expressions for Lie derivatives of non-commuting scalars, contravariant vectors and gradients. With the help of these expressions we reformulate classical canonical mechanics without assuming that the phase space variables commute. It then turns out that this is possible only if they commute, or if they satisfy the Heisenberg commutation relations.

2. *Canonical Structures*

Basic to the canonical description of classical mechanical systems of n degrees of freedom is a $2n$-dimensional affine space, $\mathscr{P}$, the phase space. One usually thinks of it as the direct sum of configuration and momentum spaces, and uses, accordingly, the symbols q^i, p_i to label its points. This attitude unduly emphasizes the distinction between positions and momenta, a distinction not preserved by the canonical group. We shall denote by z^α, where $\alpha, \beta \ldots = 1, \ldots, 2n$, the coordinates of the points of phase space.

The observables of classical mechanics are represented by real-valued functions in phase space. They naturally form an associative and commutative algebra, $\mathscr{A}$. Thus, $\mathscr{A}$ is a linear space over the field of real numbers, and is closed under multiplication of functions.

The basic postulate of canonical mechanics introduces, in addition, a Lie structure into this algebra, namely the Poisson bracket. It is convenient to define the Poisson product, $\overleftrightarrow{\nabla}$, by the identity $f\overleftrightarrow{\nabla}g \equiv \{f, g\}$, where $f, g \in \mathscr{A}$. Thus, $\overleftrightarrow{\nabla} = \overleftarrow{\partial}_\alpha \epsilon^{\alpha\beta} \overrightarrow{\partial}_\beta$, where $\partial_\alpha \stackrel{\text{Def}}{=} \partial/\partial z^\alpha$, and also $f_{,\alpha} \stackrel{\text{Def}}{=} \partial_\alpha f$. We use the summation convention over repeated indices of opposite variance. The symbols $\epsilon^{\alpha\beta}$ represent the components of the *fundamental symplectic tensor* of phase space in contravariant form. If one takes $z^i = p_i$, $z^{n+i} = q^i$, it reads

$$(\epsilon^{\alpha\beta}) = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

In general, it is an arbitrarily given, regular antisymmetric tensor. If its components are not constants, the Jacobi identity implies that its exterior derivative must vanish.

The covariant form, $\epsilon_{\alpha\beta}$, of the symplectic tensor is defined by the relation

$$\epsilon^{\alpha\gamma}\epsilon_{\beta\gamma} = \delta^{\alpha}_{\beta} \tag{2.1}$$

Indices are raised and lowered according to the rules: $\epsilon^{\alpha\beta}T_{\beta} = T^{\alpha}$, $T^{\alpha}\epsilon_{\alpha\beta} = T_{\beta}$. The sign of the symplectic scalar product depends on the relative positions of the contracted indices, e.g. $a^{\alpha}b_{\alpha} = -a_{\alpha}b^{\alpha}$. Thus, the Poisson product assumes the simple form:

$$\overleftrightarrow{\nabla} = \overleftarrow{\partial}_{\rho}\overrightarrow{\partial}^{\rho} = -\overleftarrow{\partial}^{\rho}\overrightarrow{\partial}_{\rho} \tag{2.2}$$

We next consider the canonical motions, i.e. the motions in phase space which preserve the canonical structure. Let $h^{\rho}(z)$ denote a vector field in $\mathscr{P}$. It generates an infinitesimal motion

$$z^{\rho} \to z^{\rho} + \delta\tau h^{\rho}(z) \tag{2.3}$$

which induces an infinitesimal motion in $\mathscr{A}$:

$$f \to f + \delta\tau h^{\rho} f_{,\rho} \tag{2.4}$$

This motion evidently preserves the associative product in $\mathscr{A}$. The requirement that it should also preserve the Lie product $f\overleftrightarrow{\nabla}g$ implies that the vector field h^{ρ} is a field of gradients:

$$h^{\rho} = h^{,\rho} \equiv \epsilon^{\rho\tau}h_{,\tau} \tag{2.5}$$

where $h(z)$ is any observable, $h \in \mathscr{A}$. It is a characteristic feature of canonical theory, that the algebra of canonical observables coincides, pointwise, with the algebra of canonical generators. Relations (2.4), (2.5) and (2.2) imply:

$$f \to f + \delta\tau f\overleftrightarrow{\nabla}h \tag{2.6}$$

Thus, the bilinear Poisson operator defines, with the observables, the infinitesimal generators of the Lie group of canonical motions.

3. *Non-commuting Variables*

We shall now develop the geometric theory of canonical mechanics outlined in Section 2 for the general case of non-commuting variables.

From now on, $\mathscr{A}$ shall denote an associative, but possibly non-commutative algebra generated by $2n$ independent generators Z^{α} over the field of complex numbers. The associative product in $\mathscr{A}$ shall be denoted by juxtaposition of elements, and the unit of the algebra by e. Thus, $ef = fe = f$ for every $f \in \mathscr{A}$. We shall use the same notations and terminology as in the case of classical mechanics. Thus, $\mathscr{P}$ shall denote the real linear space spanned by the elements Z^{α}, and we shall call it the *phase space*. The elements $f \in \mathscr{A}$ shall be called *functions* of the variables Z^α. For the purpose of this paper, all calculations can be performed in the subalgebra of polynomials, i.e. with elements of the form:

$$f(Z) = f_0 + f_\rho Z^\rho + f_{\rho\tau} Z^\rho Z^\tau + \dots$$

where the coefficients $f_{\rho \dots \sigma}$ are complex numbers.

The commutation relations of the algebra $\mathscr{A}$ are not specified *a priori*. They are implied by the requirement that $\mathscr{A}$ should admit a canonical structure.

In the geometric approach we are investigating in this paper, canonical mechanics is formulated in terms of quantities which we call tensors (only scalars and vectors are needed as fields), in analogy with the commutative case. The components of these tensors are functions of the variables Z^α, and their transformations are defined by the admissible transformations of coordinates in $\mathscr{P}$. The latter are necessarily linear, i.e.

$$Z^\rho \to Z^{\rho'} = C^{\rho'}_\tau Z^\tau, \qquad \mathrm{Det}(C^{\rho'}_\tau) \neq 0 \tag{3.1}$$

Scalars and vectors are defined by their usual transformation properties:

$$f'(Z') = f(Z), \qquad f^{\rho'}(Z') = C^{\rho'}_\tau f^\tau(Z) \tag{3.2}$$

where, for each value of $\rho, f^\rho(Z) \in \mathscr{A}$.

4. *Polarizations*

In this section we introduce the basic differential operator of non-commutative analysis. In ordinary analysis one considers the partial derivatives as the basic differential operators. The other differential operators of tensor analysis are then defined in terms of partial derivatives. This approach does not work in the non-commutative case. We shall see that in this case one must consider the directional derivative, or polarization, as the basic operator. The gradient appears as a simple case of polarization, while the Lie derivatives must be carefully defined. This will be done in the next section.

In ordinary analysis, the directional derivative of a function $f(z)$ in the direction of a given vector field $h^\rho(z)$ is $h^\rho f_{,\rho}$. In the non-commutative case we shall have to retain the structural source of this expression as the definition of the directional derivative. To this purpose, let a contravariant vector $h^\rho(Z)$ be given. It generates an infinitesimal motion in $\mathscr{A}$ defined by the mapping

$$f(Z) \to \bar{f}(Z) \overset{\mathrm{Def}}{=} f(Z + \delta\tau h) \tag{4.1}$$

We shall call $\bar{f}(Z)$ the *varied* function. For the variables Z^ρ themselves, the mapping (4.1) reads

$$Z^\rho \to \bar{Z}^\rho(Z) = Z^\rho + \delta\tau h^\rho(Z) \tag{4.2}$$

Unless $h^\rho(Z)$ is a linear function of the variables Z, the phase space is not stable under the mapping (4.2), i.e. $\bar{Z}^\rho \notin \mathscr{P}$.

In the commutative case, one can interpret the motion defined by relation (2.3) as either a motion of phase space into itself, or as a variation of the identity function, i.e. of the function $I: Z^{\rho} \to Z^{\rho}$. For non-linear motions, only the latter interpretation is possible in the non-commutative case. Hence, every non-linear motion is necessarily a motion in $\mathscr{A}$, as illustrated in Fig. 1.

The unit e is a fixed point with respect to all motions, and so are the points of the one-dimensional linear subspace $\mathbb{C}e \subset \mathscr{A}$, where $\mathbb{C}$ is the field of complex numbers. There are no other fixed points.

The directional differential of a function is the difference

$$\delta f = f(Z + \delta\tau h(Z)) - f(Z)$$

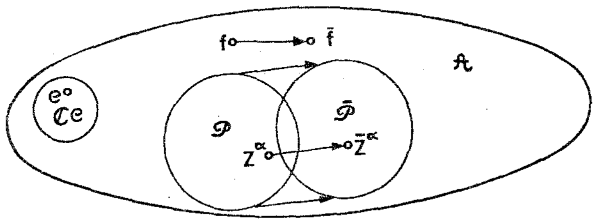


Figure 1

between the varied function and the original function. The corresponding directional derivative shall be denoted by $\mathscr{P}_{h\cdot} f$, and defined by the relation:

$$\mathscr{P}_{h\cdot} f \overset{\text{Def}}{=} \left[\frac{d}{d\tau} f(Z + \tau h(Z))\right]_{\tau=0} = \delta f / \delta\tau \tag{4.3}$$

The linear operator $\mathscr{P}_{h\cdot}$ is called the *polarization operator*, and the vector h^{ρ}, the *polarization vector*. Relation (4.3) defines the result of polarization as the coefficient of the linear term in τ in the polynomial expansion of the function $f(Z + \tau h(Z))$.

The process of polarization of a polynomial f consists in successively substituting the component h^{ρ} of the polarization vector at every occurrence of the symbol Z^{ρ} in $f(Z)$ and summing all terms thus obtained. For example:

$$\mathscr{P}_{h\cdot}(f_{\rho\tau} Z^{\rho} Z^{\tau}) = f_{\rho\tau} h^{\rho}(Z) Z^{\tau} + f_{\rho\tau} Z^{\rho} h^{\tau}(Z)$$

In the commutative case, one can write:

$$\mathscr{P}_{h\cdot} = h^{\rho}\, \partial_{\rho} \tag{4.4}$$

One directly verifies that in the non-commutative case the polarization defined by relation (4.3) is a derivation, i.e. it satisfies the identity:

$$\mathscr{P}_{h\cdot}(fg) = (\mathscr{P}_{h\cdot} f) g + f (\mathscr{P}_{h\cdot} g) \tag{4.5}$$

while the operator $h^\rho \partial_\rho$ does not. One also verifies the following expression for the commutator of two polarizations:

$$[\mathscr{P}_{h^{\cdot}}, \mathscr{P}_{f^{\cdot}}] = \mathscr{P}_{\langle f^{\cdot}, h^{\cdot} \rangle} \tag{4.6}$$

where the angular bracket is defined by the relation:

$$\langle f^\rho, h^{\cdot} \rangle \overset{\text{Def}}{=} \mathscr{P}_{h^{\cdot}} f^\rho - \mathscr{P}_{f^{\cdot}} h^\rho \tag{4.7}$$

The index of the polarization vector shall be called the *polarization index*. Since it is a bound index, in the sense that it is not free in the result of polarization, we indicate it by a dot. This convention enables us to express easily polarizations with respect to tensors, where the polarization index is indicated by a dot, the other indices remaining unaffected.

We shall need, in this work, only two polarization tensors: The Kronecker tensor, δ^α_β, and the symplectic tensor $\epsilon^{\alpha\beta}$, $\epsilon_{\alpha\beta}$. The corresponding polarizations are called *partial derivatives*. We denote them by the symbol $\mathscr{D}_\rho$:

$$\mathscr{D}_\rho \overset{\text{Def}}{=} \mathscr{P}_{\delta^{\cdot}_\rho}, \qquad \mathscr{D}^\rho \overset{\text{Def}}{=} \mathscr{P}_{\epsilon^{\rho\cdot}}. \tag{4.8}$$

One easily verifies the commutation relation $[\mathscr{D}_\rho, \mathscr{D}_\tau] = 0$. If the symplectic tensor is constant, as it is in our case, one also has:

$$\mathscr{D}^\rho = \epsilon^{\rho\tau} \mathscr{D}_\tau \tag{4.9}$$

and the contravariant operators commute. Otherwise they do not. The usual conventions $f_{,\rho} = \mathscr{D}_\rho f$, $f^{,\rho} = \mathscr{D}^\rho f$, shall be used.

5. *Lie Derivatives*

In this section we shall derive expressions for Lie derivatives which are sufficiently abstract to be meaningful in the case of non-commuting variables, and which agree with the ordinary definitions if the variables commute.

Intuitively, the Lie derivative of a geometric object measures the difference between the varied object and the transported object, both concepts being properly defined if the polarization vector is given.

If $T(Z)$ represents any object, the varied object is defined by the following relation:

$$\bar{T}(Z) \overset{\text{Def}}{=} T(Z + \delta\tau h(Z)) = T(Z) + \delta\tau \mathscr{P}_{h^{\cdot}} T(Z) \tag{5.1}$$

Variation is a motion in $\mathscr{A}$, as illustrated in Fig. 1.

The transported object is the object itself, appropriately transported to the new point. The algorithmic definition of transport depends on the nature of the object.

We shall use the symbol $\mathscr{L}_{h^{\cdot}}$ for the Lie operator with respect to a polarization vector $h^\rho(Z)$. Hence, we adopt the following definition for the Lie derivative of an object T, $\tilde{T}$ being the transported object.

$$\delta\tau \mathscr{L}_{h^{\cdot}} T \overset{\text{Def}}{=} \bar{T} - \tilde{T} \tag{5.2}$$

It is clear that if an object has a vanishing Lie derivative with respect to the polarization vector defining some group of motions, it is invariant under the motions of this group.

In the case of commuting variables, one can obtain the Lie derivatives of any type of tensor from the Lie derivatives of three basic types; namely the scalars and the two types of vectors, by requiring that the Lie derivative of a general tensor be a linear operation, that it commute with contraction of indices, and that it be a derivation with respect to the tensor product. The extension of the Lie derivatives to non-commuting tensors shall not be attempted, since the only Lie derivatives needed in canonical mechanics are those of scalars, contravariant vectors and gradients of scalars, the components of these objects being functions of the Z's. In addition, we shall use the Lie derivatives of constant tensors. They can be taken from the classical theory.

5.1. *Scalars*

Let $\mathscr{S}$ denote the set of scalars. Pointwise, $\mathscr{S} = \mathscr{A}$. For any $f \in \mathscr{S}$, the expression for the varied scalar is given by relation (5.1) and the substitution $T = f$. In the commutative case, the value of a scalar at a point of phase space is a number, and one transports a number by assigning it to the new point. Hence, the transported scalar is defined by the relation $\tilde{f}(z + \delta\tau h(z)) = f(z)$. By adopting the same formal definition in the case of non-commuting variables, one obtains, from relation (5.2), the expression:

$$\mathscr{L}_{h \cdot} f = \mathscr{P}_{h \cdot} f \tag{5.1.1}$$

While the polarization is defined for any function, the Lie derivative is a structure-preserving operator, i.e. it maps tensors into tensors of the same type. Hence, relation (5.1.1) is valid only for scalars.

5.2. *Contravariant Vectors*

The set of contravariant vectors shall be denoted by $\mathscr{V}$. For any $f^\rho \in \mathscr{V}$, the varied vector is defined by relation (5.1). We must now find the correct definition for the transported vector.

In the commutative case, $f^\rho(z)$ belongs to the tangent space at the point $z^\alpha \in \mathscr{P}$. One can obtain the definition of its transport by studying the motion induced in the tangent space by a motion of the manifold $\mathscr{P}$. This idea is inapplicable in the non-commutative case, but it suggests the following similar process: Consider the pair of infinitesimally near spaces $\mathscr{P}, \hat{\mathscr{P}} \subset \mathscr{A}$, linearly spanned, respectively, by the sets of variables Z^ρ and $\hat{Z}^\rho = Z^\rho + \delta\sigma f^\rho(Z)$. Conversely, the pair of spaces $\mathscr{P}, \hat{\mathscr{P}}$, uniquely defines the vector $f^\rho(Z)$. An infinitesimal motion defined by the polarization vector $h(Z)$ maps this pair of adjacent spaces onto the pair of adjacent spaces $\mathscr{P}', \hat{\mathscr{P}}' \subset \mathscr{A}$. The vector $\tilde{f}^\rho(Z')$ defined by this new pair of spaces is the transported vector. The construction is illustrated in Fig. 2. By performing the substitutions indicated in the diagram, one obtains the following expression for the Lie derivative:

$$\mathscr{L}_{h^{\cdot}} f^{\rho} = \mathscr{P}_{h^{\cdot}} f^{\rho} - \mathscr{P}_{f^{\cdot}} h^{\rho} \tag{5.2.1}$$

If the variables commute, relation (4.4) holds, and this expression assumes its usual form:

$$\mathscr{L}_{h^{\cdot}} f^{\rho} = h^{\tau} f^{\rho}{}_{,\tau} - f^{\tau} h^{\rho}{}_{,\tau} \tag{5.2.2}$$

Relation (5.2.1) has already appeared in a different role, relation (4.7). Comparison of these two equations yields the following relation between the symbols corresponding to the two roles of the Lie derivative:

$$\langle f^{\rho}, h^{\cdot} \rangle = \mathscr{L}_{h^{\cdot}} f^{\rho} \tag{5.2.3}$$

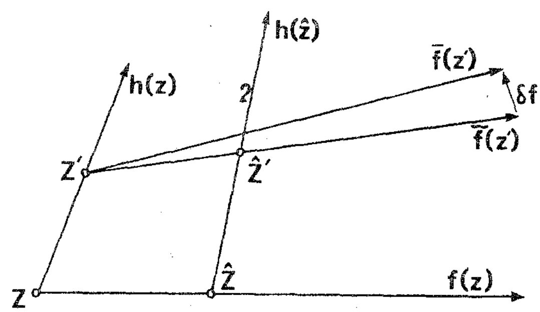


Figure 2

The angular bracket is the *Lie bracket*. In commutative analysis it is denoted by square brackets, but we reserve this symbol for the algebraic commutator. The Lie bracket is obviously bilinear and antisymmetric.

By straightforward substitution of relations (4.6) and (4.7) one verifies the validity of the Jacobi identity:

$$\langle\langle f^{\rho}, h^{\cdot}\rangle, g^{\cdot}\rangle + \langle\langle g^{\rho}, f^{\cdot}\rangle, h^{\cdot}\rangle + \langle\langle h^{\rho}, g^{\cdot}\rangle, f^{\cdot}\rangle = 0 \tag{5.2.4}$$

Hence, the linear space $\mathscr{V}$ of polarization vectors is a Lie algebra with the bracket (5.2.3) as Lie product.

From relations (5.2.3) and (5.2.4) one obtains the following expression for the commutator of Lie derivatives of contravariant vectors, g^{ρ}.

$$[\mathscr{L}_{h^{\cdot}}, \mathscr{L}_{f^{\cdot}}] g^{\rho} = \mathscr{L}_{\langle f^{\cdot}, h^{\cdot}\rangle} g^{\rho} \tag{5.2.5}$$

5.3. *Gradients*

The set of gradients shall be denoted by $\mathscr{G}$. For any scalar $f(Z) \in \mathscr{S}$, we are to find the Lie derivative of the gradient $f_{,\rho} \in \mathscr{G}$, defined by relation

(4.8). The calculation is most transparent if done at the point $Z^{\alpha} - \delta\tau h^{\alpha}(Z)$. The defining relation for the Lie derivative is then:

$$\delta\tau \mathscr{L}_{h\cdot} f_{,\rho}(Z - \delta\tau h) = \tilde{f}_{,\rho}(Z - \delta\tau h) - \bar{f}_{,\rho}(Z - \delta\tau h) \tag{5.3.1}$$

Relation (5.1) yields $\tilde{f}_{,\rho}(Z - \delta\tau h) = f_{,\rho}(Z)$ for the varied gradient. The transported gradient is naturally defined by the relation $\bar{f}_{,\rho}(Z - \delta\tau h) = \mathscr{D}_{\rho} f(Z - \delta\tau h)$. Disregarding the $\delta\tau^2$ terms in the expansion of the left-hand side of relation (5.3.1), one obtains:

$$\mathscr{L}_{h\cdot} f_{,\rho}(Z) = \mathscr{D}_{\rho} \mathscr{P}_{h\cdot} f(Z) \tag{5.3.2}$$

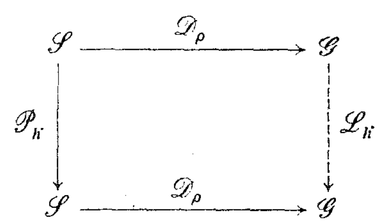
One easily verifies that the operator $\mathscr{D}_{\rho}$ can be distributed over the functions f and h^{α} according to the derivation rule. One thus obtains the expression

$$\mathscr{L}_{h\cdot} f_{,\rho} = \mathscr{P}_{h\cdot,\rho} f + \mathscr{P}_{h\cdot} f_{,\rho} \tag{5.3.3}$$

In the case of commuting variables, for which relation (4.4) holds' expression (5.3.3) assumes the standard form

$$\mathscr{L}_{h\cdot} f_{,\rho} = h^{\tau}{}_{,\rho} f_{,\tau} + h^{\tau} f_{,\rho\tau} \tag{5.3.4}$$

According to relation (5.3.2), the Lie derivative of gradients is the linear operator $\mathscr{L}_{h\cdot}$ which makes the following diagram commute:



5.4. *Central Tensors*

A central object is an object whose components belong to the centre of the algebra $\mathscr{A}$. If either the polarization vector or the object to which the Lie derivative is applied is central, the Lie derivative assumes its classical expression, since relation (4.4) applies. This fact has already been used in the definition of the gradient, which is a polarization with respect to a central object, relation (4.8). We list here, for future reference, the expressions for the Lie derivatives of some central tensors, C. The polarization vector is general.

$$\mathscr{L}_{h\cdot} C^{\alpha\beta} = h^{\rho} C^{\alpha\beta}{}_{,\rho} - C^{\rho\beta} h^{\alpha}{}_{,\rho} - C^{\alpha\rho} h^{\beta}{}_{,\rho} \tag{5.4.1}$$

$$\mathscr{L}_{h\cdot} C^{\alpha}{}_{\beta} = h^{\rho} C^{\alpha}{}_{\beta,\rho} - C^{\rho}{}_{\beta} h^{\alpha}{}_{,\rho} + C^{\alpha}{}_{\rho} h^{\rho}{}_{,\beta} \tag{5.4.2}$$

$$\mathscr{L}_{h\cdot} C_{\alpha\beta} = h^{\rho} C_{\alpha\beta,\rho} + C_{\rho\beta} h^{\rho}{}_{,\alpha} + C_{\alpha\rho} h^{\rho}{}_{,\beta} \tag{5.4.3}$$

If the algebra is central, the first terms on the right-hand sides of these relations vanish.

5.5. General Properties

Let $\mathscr{T}$ denote any of the sets of objects considered so far, i.e. $\mathscr{S}$, $\mathscr{V}$, $\mathscr{G}$, or the central tensors. Relation (4.3) obviously implies that the polarization operator is bilinear, i.e. linear in the object, and linear in the polarization vector. Thus, the Lie derivatives can be viewed as bilinear operators:

$$\mathscr{P}: \mathscr{T} \otimes \mathscr{V} \to \mathscr{T}$$

$$\mathscr{L}: \mathscr{T} \otimes \mathscr{V} \to \mathscr{T}$$

Relation (5.2.1) implies that the Lie derivative of contravariant vectors is antisymmetric:

$$\mathscr{L}: \mathscr{V} \wedge \mathscr{V} \to \mathscr{V}$$

It is convenient to introduce the concept of *Lie motion* in the spaces $\mathscr{T}$, an infinitesimal Lie motion being defined by the operator

$$L_{h\cdot} \overset{\text{Def}}{=} I + \delta\tau \mathscr{L}_{h\cdot} \tag{5.5.1}$$

which maps $\mathscr{T}$ into itself.

The invariance group of an object $T \in \mathscr{T}$ is defined by the Lie subalgebra $\mathscr{V}_T \subset \mathscr{V}$ of generators satisfying the requirements of conservation and stability:

$$\mathscr{L}: T \times \mathscr{V}_T \to \{0\}$$

$$\mathscr{L}: \mathscr{V}_T \wedge \mathscr{V}_T \to \mathscr{V}_T$$

Thus, the conserved objects of a group of motions are the fixed points of the corresponding group of Lie motions. For example, if the object T represents the metric tensor of the space, the elements of $\mathscr{V}_T$ are the *Killing vectors*. The corresponding group of motions is the *symmetry group* of the space.

6. *Canonical Algebras*

The fundamental, or metric tensor of canonical mechanics is the symplectic tensor $\epsilon^{\rho\tau}$. The elements of the Lie algebra of generators $h^\rho \in \mathscr{V}_T$, for $T = \epsilon^{\rho\tau}$, are the solutions of the Killing condition

$$\mathscr{L}_{h\cdot} \epsilon^{\rho\tau} = 0 \tag{6.1}$$

Substitution of $\epsilon^{\rho\tau}$ for $C^{\rho\tau}$ in relation (5.4.1) yields $h^{\rho\cdot\tau} = h^{\tau\cdot\rho}$. This relation implies that the Killing vectors are gradients:

$$h^\rho = h^{\cdot\rho} \tag{6.2}$$

where $h(Z)$ is any point of $\mathscr{A}$. This is, formally, the same solution as in classical mechanics, relation (2.5).

The Lie bracket of gradients shall be denoted by a bracket of scalar generators:

$$\langle f, g \rangle \overset{\text{Def}}{=} \langle f^{\cdot\rho}, g^{\cdot\cdot} \rangle$$

We shall call it the *canonical product* in $\mathscr{A}$. By relation (5.2.4), it is a Lie product. The Lie derivatives with respect to gradients shall be called *canonical derivatives*, and shall be denoted by $\mathscr{L}_h$. A canonical derivative applied to a scalar is the same as a *canonical polarization*:

$$\mathscr{P}_h f = \langle f, h \rangle \tag{6.3}$$

Substitution of the central tensor δ^α_β into relation (5.4.2) yields $\mathscr{L}_h . \delta^\alpha_\beta = 0$, which means that contractions inside mixed tensors commute with the Lie derivative if the latter is defined. Applied to $C_{\alpha\beta} = \epsilon_{\alpha\beta}$, relation (5.4.3) also yields the gradients.

The results obtained so far are independent of the commutation properties of the algebra $\mathscr{A}$. The space $\mathscr{V}$ of polarization vectors defines in the space of objects two continuous groups: The group of motions, the neighborhood of whose identity consists of the operators

$$P_h . \stackrel{\text{Def}}{=} I + \delta\tau \mathscr{P}_h . \tag{6.4}$$

and the group of Lie motions, the neighborhood of whose identity consists of the operators $L_h .$ defined by relation (5.5.1). If $\mathscr{T} = \mathscr{A}$, these two groups coincide.

The group of motions is also the group of automorphisms of the associative algebraic substructure of $\mathscr{A}$, i.e. the following mapping diagram commutes

$$\begin{array}{ccccc}
\mathscr{A} & \otimes & \mathscr{A} & \xrightarrow{\ \cdot\ } & \mathscr{A} \\
\downarrow P_h . & & \downarrow P_h . & & \downarrow P_h . \\
\mathscr{A} & \otimes & \mathscr{A} & \xrightarrow{\ \cdot\ } & \mathscr{A}
\end{array}$$

This diagram is equivalent to the derivation property, (4.5), of the polarization operator.

Of the three spaces $\mathscr{T}$ considered in canonical mechanics, $\mathscr{S}$ i.e. $\mathscr{A}$, is the only one equipped with an associative algebraic product. Thus, the automorphism diagram shown above is the only such diagram in the theory. Moreover, the Lie algebra $\mathscr{V}$ of vector generators of motions is independent of the structure of the space $\mathscr{T}$ in which it acts. This is not so for the group generated by the Killing fields. The generators are now observables themselves, and their Lie structure must be consistent with their associative structure. This consistency requirement imposes conditions on the commutation properties of the algebra $\mathscr{A}$. The derivation property must be bilateral, i.e. the canonical polarization must be a derivation with respect to the product of dynamical variables to which it is applied, as well as to the product of generators. This imposes the following commutation relations on the variables Z^ρ:

$$[Z^\rho, Z^\tau] = K\epsilon^{\rho\tau} \tag{6.5}$$

where K is any central element. If the algebra is central, K is a complex number. In the usual representation of the variables Z^α by Hermitian operators, $K = ih$.

In order to prove relation (6.5), one expands the canonical products in the identity $\langle fg, hk\rangle \equiv \langle hk, fg\rangle$ according to the derivation rule, first to the right, and then to the left. The resulting identity reads:

$$\langle f, h\rangle [g, k] \equiv [f, h] \langle g, k\rangle \tag{6.6}$$

This identity can be satisfied only if

$$[g, h] = K\langle g, h\rangle \tag{6.7}$$

and one verifies by further substitutions that K commutes with all elements of $\mathscr{A}$. Substitution of $g = r_\rho Z^\rho$, $h = s_\rho Z^\rho$, into relation (6.3) yields $\langle g, h\rangle = \epsilon^{\rho\tau} r_\rho s_\tau$, which proves relation (6.5).

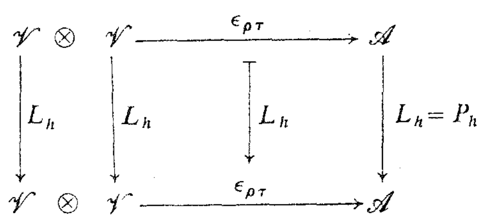
Relation (6.5) characterizes all associative algebras which admit a canonical structure. For $K = 0$ the algebra is commutative, and one verifies that the canonical product is the Poisson bracket. If $K = \text{const.} \neq 0$, relation (6.5) is the Heisenberg commutation rule. In this case, relation (6.7) expresses the equivalence of canonical Lie products and algebraic commutators.

7. *The Scalar Product*

In this section we consider the symplectic scalar product of contravariant vectors, which is, in classical mechanics, fundamentally related to the states.

For any two infinitesimal vectors $f^\rho(z)d\tau$, $g^\sigma(z)\delta\tau$, in classical phase space, the scalar product $\epsilon_{\rho\tau} f^\rho g^\tau d\tau\, \delta\tau$ is the measure of the infinitesimal parallelogram they span.

In the general case, the algebra of generators of the invariance group of the scalar product is the set of functions $h(Z)$ for which the following diagram is commutative



The expansion of this diagram yields the relation

$$(\mathscr{P}_f . h_{,\tau}) g^\tau - f^\tau (\mathscr{P}_g . h_{,\tau}) = 0 \tag{7.1}$$

which is to be interpreted as an identity in the vectors f^ρ, g^τ, and as a differential condition on the scalar generator $h(Z)$.

In the case of classical mechanics, where relation (4.4) holds, one verifies that all points $h \in \mathscr{A}$ satisfy relation (7.1), i.e. the invariance group of the scalar product is the canonical group itself. This is essentially Liouville's theorem.

In the case of quantum mechanics, one verifies that only polynomials at most quadratic in the variables Z^α satisfy relation (7.1). Due to relation (6.5), the coefficients of the quadratic terms of these polynomials are symmetric, and thus, the invariance group of the scalar product is the inhomogeneous symplectic group. The scalar product thus intrinsically distinguishes the symplectic subgroup of the canonical group in the case of quantum mechanics.

References

Falk, G. (1951). *Mathematische Annalen*, **123**, 379.

Grgin, E. and Petersen, A. (1972). 'Classical and quantum mechanics in auxiliary algebras.' *Physical Review* **D5**, 300.